Luice de-Broglie hypothesis :⇒

According to quantum theory the radiation (wave) exhibit particle like properties. In 1925 de-Broglie suggested that all material particles have characteristics of wave. The wavelength λ of the wave associated with the particle is given by

$$\boxed{\lambda = \frac{h}{p}}$$ where h = Planck's constant. p = momentum = mv

This relation is know as de-Broglie relation.

de-Broglie was trying to discover the underline significance of Bohr's quantization condition, this idea was to fit an integral no. of standing waves in analogy with the integral no. of waves (or half waves) in a string stretched between two rigidly fixed ends.

Thus if r_n be the radius of n^{th} bohr orbit then according to de-Broglie the perimeter of the orbit ~~on~~ ie $2\pi r_n$ must obey the condition which is — $2\pi r_n = n\lambda$ or $2\pi r_n = n\lambda$ ... (1)

where n is an integral no. in order that an integral no. of waves may be fitted into the orbit.

From Bohr quantization condition we have —

$$m v r_n = \frac{nh}{2\pi}$$

Putting the value of r_n from the equation no (1) we get —

$$m v \cdot \frac{n\lambda}{2\pi} = \frac{nh}{2\pi} \quad \text{or} \quad m v \lambda = h$$

$$\text{or} \quad p\lambda = h \quad \text{or} \quad \boxed{\lambda = \frac{h}{p}}$$

This is the de-Broglie hypothesis.

particle of energy E and momentum p may exhibit characteristics of wave of wavelength λ as given above and its freq. is determined by the well known plancks formula $E = h\nu$.

So, quantum mechanically the momentum and energy can be expressed as –

$$p = \frac{h}{\lambda} = \frac{h/2\pi}{\lambda/2\pi} = \hbar k.$$

$$\text{and} \quad E = h\nu = \hbar\omega.$$

where $k = \frac{2\pi}{\lambda}$ is the magnitude of the propagation vector and $\omega = 2\pi\nu$ is the circular freq.

classically a wave of definite freq. and wavelength is of infinite extent in space (spatial) and of infinite duration (Temporal extension)

on the other hand a corpuscle is localized at a definite point in space at a given instant of time and has a definite momentum p and energy E.

Thus the basic characteristics of wave and corpuscle is are incompatible.

In order to bring about reconcilation between wave and corpuscular characters viewpoint, we have to localize the wave in a finite region of space. This can be achived by the superposition of waves of different wavelengths upon one another by the well known method of Fourier transform.

Heisenberg uncertanity principle: →

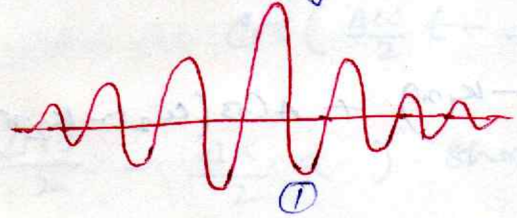
This principle states that both the position and momentum of a particle can not be measured simultanously with any desired degree of accuracy.

If Δx be the uncertanity in the measurement of position and Δp_x be the uncertainity in uncertainity principle —

$$\Delta x \Delta p_x \geq \hbar/2 \quad \Big(\text{where } \hbar = h/2\pi .$$

Wave packet and the origin of uncertainity principle—

The fact that a moving body may be regarded as a de-Broglie wave group rather than as a localized entity suggest that there is a fundamental limit to the accuracy with which we can measure its particle properties. The figure shown below shows de-Broglie wave group.



①

②

The particle may be anywhere within the wave group. If the wave group is made very narrow as shown in fig. ② the position of the particle can be measured more accurately, but it can be shown that the constituents waves of this group have a large no. of variation in the wavelength λ (or in the propogation constant $K = \frac{2\pi}{\lambda}$). This means that the momentum $p = \frac{h}{\lambda}$ of the particle becomes more inaccurate or uncertain. Thus there must be a certain relation connecting the inherent uncertainity Δx in the measurement of particle position to the inherent uncertainity Δp in the simultaneous measurement of particle momentum.

According to Heisenberg the relation connecting the inherent uncertainty Δx in position and the in uncertainity Δp in momentum is —

$$\Delta x \Delta p \geq \hbar/2$$

where $\hbar = h/2\pi$ is the planck's constant.

To obtain the relation we consider a wave group formed by two only two components of waves having equal amplitude A and freq.s ω_1 and ω_2 differing by a small amount and represented by the equation

$$\psi_1 = A\cos(\omega_1 t - k_1 x)$$

$$\psi_2 = A\cos(\omega_2 t - k_2 x).$$

Superposition gives the resultant wave which is —

$$\psi = \psi_1 + \psi_2$$

$$= A\cos(\omega_1 t - k_1 x) + A\cos(\omega_2 t - k_2 x).$$

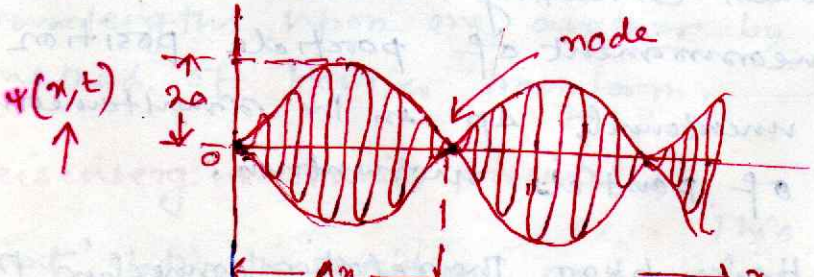
or $\psi = 2A\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)\cos(\omega t - kx)$

where we put $\omega = \frac{\omega_1+\omega_2}{2}$, $k = \frac{k_1+k_2}{2}$

$$\Delta\omega = \omega_1 - \omega_2 \quad \text{and} \quad \Delta k = k_1 - k_2$$

We see that the above equation represents a wave having an envelope of amplitude equal to $2A\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)$ ~~mod~~ modulating the cosine wave $\cos(\omega t - kx)$.

The resultant wave is shown in the following fig.



The loop or the wave packet formed by the ~~sup~~ component wave travell with a velocity

$$v_g = \frac{\Delta\omega}{\Delta k}.$$

As the group velocity is equal to the particle velocity, the loop formed is equivalent to the particle position. Thus the particle's position can not be given with certainity. It lies somewhere in between the two consequitive nodes. In other words the uncertainity in the measurement of the position of the particle is equal to the distance between two consequitive nodes.

The condition for the formation of a node is -

$$\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta K}{2}x\right) = 0,$$

$\left(\frac{\Delta\omega}{2} - \frac{\Delta K}{2}x\right)$ should be odd multiple of $\pi/2$

ie $\frac{\Delta\omega}{2} - \frac{\Delta K}{2}x = \pi/2,\ 3\pi/2 \ldots (2n+1)\pi/2$

where $n = 0, 1, 2, \ldots$

If x_1 and x_2 be the positions of two consequitive nodes we must have, -

$$\frac{\Delta\omega}{2}t - \frac{\Delta K}{2}x_1 = (2n+1)\pi/2$$

and $$\frac{\Delta\omega}{2}t - \frac{\Delta K}{2}x_2 = (2n+3)\pi/2$$

$\therefore\ \frac{\Delta K}{2}(x_1 - x_2) = \pi$ or $x_1 - x_2 = \frac{2\pi}{\Delta K}$

$\boxed{\Delta x \Delta K = 2\pi}$ where $\Delta x = (x_1 - x_2)$.

Again de-Broglie relation -

$$\lambda = \frac{h}{p}.$$

$$\therefore\ K = \frac{2\pi}{\lambda} = \frac{2\pi p}{h}.$$

$$\text{or } \Delta K = \frac{2\pi}{h}\Delta p.$$

Thus, $\Delta x \Delta K = \Delta x \cdot \frac{2\pi}{h}\Delta p = 2\pi$

$\boxed{\Delta x \Delta p \simeq h}$

where Δp is the uncertainty in the simultaneous measurement of momentum.

If we consider a very large no. of ~~con~~ component waves then of continuously varying freq. we will get — $\Delta x \Delta p \geq \hbar/2$

For the different co-ordinate direction we have

$$\Delta y \Delta p_y \geq \hbar/2 \quad \text{and} \quad \Delta z \Delta p_z \geq \hbar/2$$

These uncertainties are not in our apparatus but present in nature.

Show that for a classical wave group ① $\Delta\omega \Delta t = 2\pi$

② $\Delta\nu \Delta t = 1$

and $\Delta E \Delta t \geq \hbar/2$

Ans. we have

$$\Delta x \Delta k = 2\pi.$$

Now we know that $k = \frac{2\pi}{\lambda} = \frac{2\pi\nu}{\lambda\nu} = \frac{\omega}{v}$.

or $\Delta k = \frac{\Delta\omega}{v}$. and $x = vt$

or $\Delta x = v\Delta t$

Thus

$$\Delta x \Delta k = v\Delta t \times \frac{\Delta\omega}{v} = \Delta\omega \Delta t = 2\pi$$

or $\boxed{\Delta x \Delta k = 2\pi}$ $\boxed{\Delta\omega \Delta t = 2\pi}$

proved

$\omega = 2\pi\nu$

or $\Delta\omega = 2\pi\Delta\nu$

$\therefore \Delta\omega\Delta t = \Delta\nu \cdot 2\pi \Delta t = 2\pi$

or $\boxed{\Delta\nu \Delta t = 1}$

To obtain the energy time uncertainty relation, we consider the classical expression —

$E = \frac{p^2}{}$

or $\Delta p = \frac{\Delta E}{v}$.

where ΔE represents the uncertainity in ~~em~~ the measurement of energy.

Again $x = vt$

or $\Delta x = v\Delta t$.

$\therefore \quad \Delta x \Delta p = v\Delta t \times \frac{\Delta E}{v} = \Delta E \Delta t \geq \hbar/2$

$$\boxed{\Delta E \Delta t \geq \hbar/2}$$

This is the time-energy uncertainity relation.

Phase velocity and group velocity of de-Broglie wave –

If the velocities of individual wave being superimposed are the same, the velocity with which the wave packet travell is the common wave velocity. However in the case of de-Broglie wave the wave velocity varies with the wavelength, and the individual wave do not travell with the same velocity. Thus the wave packet has a different velocity from the waves that composes it.

Phase velocity v_p is given by

$$v_p = \frac{\omega}{K}.$$

Now from de-Broglie hypothesis

$$\lambda = \frac{h}{p} \quad \text{or} \quad p = \frac{h}{\lambda} = \frac{h/2\pi}{\lambda/2\pi} = \hbar K.$$

and from Einstein relation

$$E = h\nu = \frac{h}{2\pi} \cdot 2\pi\nu = \hbar\omega.$$

Thus $\frac{E}{p} = \frac{\hbar\omega}{\hbar K} = \frac{\omega}{K} = v_p$

$$\therefore \quad v_p = \frac{E}{p}.$$

For non-relativistic domain $\Rightarrow$

when the particle velocity v is small compared to c, the

$$E = \frac{p^2}{2m}$$

$$\therefore v_p = \frac{E}{p} = \frac{p^2}{2mp} = \frac{p}{2m} = \frac{mv}{2m} = \frac{v}{2}$$

This results seems disturbing because it appears that the matter wave would not be able to keep up with the particle whose motion it controls.

For relativistic domain :⇒

In this case

$$E = mc^2$$

$$v_p = \frac{E}{p} = \frac{mc^2}{mv} = \frac{c^2}{v}$$

Since $v < c$ always we must have $v_p > c$.
Therefore phase velocity is greater than the velocity ~~in~~ of light in vacuum.

This will not violate the Einstein postulate of special theory of relativity because no physical quantity like energy, information or signals etc associated with the wave can travel with the phase velocity. The phase velocity is the velocity of propagation of the phase of the disturbance.

From this we conclude that no particle can be expressed ~~as~~ by a single or monochromatic wave. The particle must be regarded as a localized entity and hence a wave packet. The wave packet is supposed to be formed by superimposing a no. of waves having wavelengths differing small in amplitude. The amplitude of the wave packet varies from zero to a certain maximum value.

Group velocity :⇒

Group velocity is defined as the velocity with which the wavepacket

$$v_g = \frac{d\omega}{dk} = \frac{d(2\pi\nu)}{d(2\pi/\lambda)} =$$

$$= \frac{d\left[2\pi\left(E/h\right)\right]}{d\left(\frac{2\pi p}{h}\right)}$$

$$= \frac{\frac{2\pi}{h}\frac{dE}{dp}}{2\pi/h} = \frac{dE}{dp}$$

$$\therefore \boxed{v_g = \frac{dE}{dp}}$$ where we use the de-Broglie relation $\lambda = \frac{h}{p}$ and the Einstein relation $E = h\nu$.

For non-relativistic domain :⇒

For a free particle of non-relativistic mass m, total energy of the particle E is given by

$$E = p^2/2m \quad \left[\text{for free particle } V=0\right].$$

$$\frac{dE}{dp} = \frac{2p}{2m} = \frac{mv}{m} = v.$$

$$\therefore \boxed{v_g = \frac{dE}{dp} = v}$$

For a relativistic domain :⇒

In this case the total relativistic energy is given by –

$$E^2 = p^2c^2 + m_0^2c^4$$

where m_0 = rest mass of the particle

Differentiating w.r.t to p we get –

$$2E\frac{dE}{dp} = 2pc^2 + 0$$

$$\text{or} \quad \frac{dE}{dp} = \frac{pc^2}{E} = \frac{mvc^2}{mc^2}$$

$$= v$$

$$\therefore \boxed{v_g = v}$$

Thus we see that whether relativistic or non-relativistic the group velocity is equal to the particle velocity.

We have $v_p = \frac{c^2}{v}$ where v = particle velocity

As we know that $v = v_g$ (group velocity)

$$\therefore v_p = \frac{c^2}{v_g}$$

$$\text{or } \boxed{v_p \cdot v_g = c^2}$$

This is the relation between phase velocity and group velocity.

Problem

The energy of a free electron including its rest mass is 1 MeV. Calculate the group and phase velocity.

Ans. $1\ eV = 1.602 \times 10^{-19}\ J$.

The relativistic relation between energy and momentum is —

$$E^2 = p^2c^2 + m_0^2c^4.$$

$$\text{or } (mc^2)^2 = p^2c^2 + m_0^2c^4$$

$$\text{or } m^2c^4 = p^2c^2 + m_0^2c^4 \text{ or } m^2c^2 = p^2 + m_0^2c^2$$

$$\text{or } p^2 = m^2c^2 - m_0^2c^2 \text{ or } p = \sqrt{m^2c^2 - m_0^2c^2}$$

$$= \sqrt{m^2 - m_0^2}\, c.$$