Quantum dynamics of a mechanical resonator coupled to a qubit

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Abstract

We have studied the quantum dynamics of a mechanical resonator coupled to a qubit. The resonator is a part of an optical cavity and is driven by external laser. The quantum Langevin operator has been derived for the Quadrature operator as well as for the fluctuation operators. We have tried to diagonalize the drift matrix to calculate the position spectrum of the resonator.

1 Introduction

Optomechanical systems can be used to study the dynamics of a mechanical resonator. The dynamics can be influenced by the presence of qubit coupled to the resonator. There are many examples of literature[1], where the dynamical behaviour of a resonator in hybrid optomechanical systems has been studied. In the reference[2], we see an investigation for the generation of entanglement between Bose-Einstein condensate(BEC) and the mirror of a hybrid optomechanical system via the cavity optical field. Ref.[3] gave a scheme for entangling an optical Fabry-Perot cavity and a nanomechanical resonator beam (NRB) to a high degree by means of quantum dot. We have tried to study here the dynamics of the resonator motion coupled to a qubit in a hybrid optomechanical system. The optomechanical coupling is due to radiation pressure of the optical field on the resonator, which couples the cavity photon number with the position of the resonator ($\propto \hat{x}\hat{n}$), where

$x = x_{zf}(b + b^{\dagger})$ represent the resonator position. $x_{zf} = \sqrt{\frac{\hbar}{2m\omega}}$($m$ is the mass and ω is the oscillation frequency of the resonator) is the zero point oscillation of the resonator,$\hat{b}$ is the phonon annihilation operator.$\hat{n} = a^{\dagger}a$ is the cavity photon number operator,$\hat{a}$ is the cavity field operator.Here We place a nanomechanical resonator(NMR)(with a quantum dot embedded on it) in an optical cavity and drive the cavity with a strong laser of frequency ω_l .The NMR motion is coupled with the cavity field due to the reason said above. The quantum dot which is taken as a two level system is also coupled with the cavity field.The quantum dot is coupled through strain mediation to the motion of the resonator. This is a hybrid optomechanical system because of

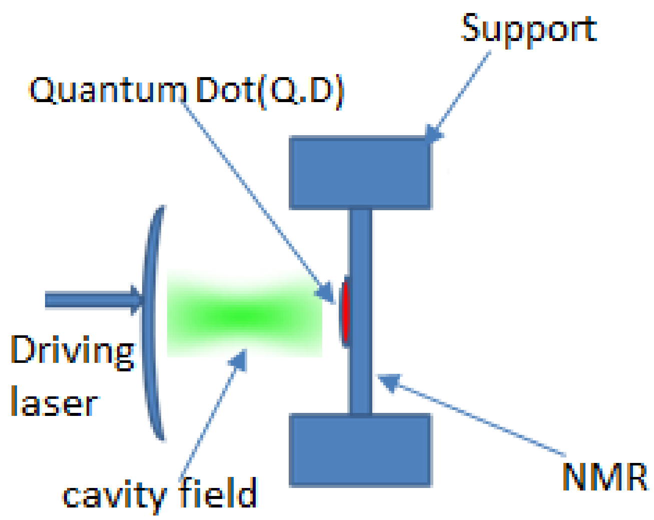


Figure 1: *Nano mechanical resonator(NMR)within an optical cavity.A Quantum dot is embedded within the cavity.A laser of frequency* ω_l *is applied to drive the cavity field*

the presence of the three different degrees of freedom namely the optical field, the mechanical motion and the Quantum dot which is taken as a two level system. In this case the doubly clamped nanomechanical resonator is acting as an end mirror of this optomechanical system and the optomechanical coupling is due to the change in momentum of the photons after reflection from the end mirror.
Similar type of system has been used to entangle a cavity optical field to a mechanical resonator by using a quantum dot[1] All the coupling can be

described clearly if we write the Hamiltonian of the system which is to be,

$$H = \hbar\omega_c a^\dagger a + \frac{\hbar\Omega_m}{2}(P^2 + Q^2) + \frac{1}{2}\hbar\omega_0\sigma_z + \hbar\lambda\sigma_z(b + b^\dagger) + \sqrt{2}\hbar g_0 a^\dagger a Q + \\ \hbar g_c(\sigma_+ a + \sigma_- a^\dagger) + i\hbar\epsilon(a^\dagger e^{-i\omega_l t} + a e^{i\omega_l t}) \tag{1}$$

Where ω_c is the cavity resonance frequency,Ω_m is the mechanical oscillation frequency and ω_0 is the quantum dot transition frequency, a and $a^\dagger$ are cavity field annihilation and creation operator.$P = \frac{b-b^\dagger}{i\sqrt{2}}, Q = \frac{b+b^\dagger}{\sqrt{2}}$ are the dimensionless momentum and position operator for the mechanical oscillator, $b, b^\dagger$ phonon annihilation and creation operator.σ_+ and σ_- are the raising and lowering operator for the dot.σ_z is the population difference operator for the dot.λ is the coupling constant between the resonator and the quantum dot. ϵ is the driving strength. In a frame rotating with the laser frequency ω_l the Hamiltonian can be written as,

$$H = \hbar\omega_c\frac{X^2+Y^2}{2} + \frac{\hbar\omega_c}{2}(P^2 + Q^2) + \frac{\hbar\omega_0\sigma_z}{2} + \sqrt{2}\hbar\lambda\sigma_z Q + \frac{1}{\sqrt{2}}\hbar g_0(X^2 + Y^2)Q + \\ \hbar g(\sigma_+\frac{X+iY}{\sqrt{2}} + \sigma_-\frac{X-iY}{i\sqrt{2}}) + \frac{1}{\sqrt{(2)}}\hbar\epsilon Y \tag{2}$$

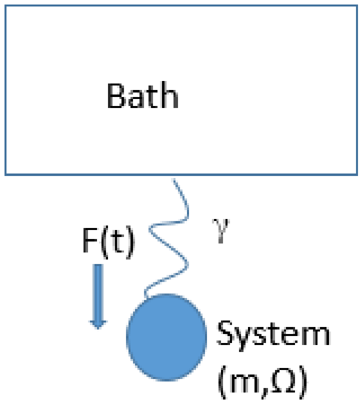
Where, $X = \frac{a+a^\dagger}{\sqrt{2}}, Y = \frac{a-a^\dagger}{i\sqrt{2}}$ are amplitude and phase Quadrature of the light field respectively.

2 Quantum Langevin Equations

The dynamics of a Quantum system coupled to its environment can be described by the method used by Langevin[1],where it is assumed that due to coupling the system dynamics is effected by the Bath whereas the bath dynamics remains almost unaffected.Several methods were developed in the 1980's and 1990's to extend the approach to open quantum processes and dynamics[2,3,4,5] motivated by the experimental progress in the field of Quantum optics. The quantum langevin equation for any system operator($\hat{O}$) is given by,

$$\hat{O} = \frac{1}{ih}[\hat{O}, \hat{H}_{sys}] - \frac{1}{ih}[\hat{O}, \hat{q}]\hat{F}(t) + \frac{m}{2i\hbar}[[\hat{O}, \hat{q}], \int_{-\infty}^{t} dt'\gamma(t-t')\dot{\hat{q}}(t')]_+ \tag{3}$$

where the anti commutator is defined by, $[A, B]_+ = AB + BA$. $\hat{q}$ is the position operator, $\hat{F}(t)$ is the operator valued stochastic force with zero expectation value i.e. $< F(t) >= 0$, which is acting on the system by the Bath.m is the mass of the system and γ represents the damping.



Now within the first Markov approximation ($\gamma(t) = \gamma\delta(t)$) the Langevin equation takes the form,

$$\begin{aligned}\dot{\hat{O}} = \tfrac{1}{i\hbar}[\hat{O},\hat{H}_{sys}] - \tfrac{1}{i\hbar}[\hat{O},\hat{q}]\hat{F}(t) \\ +\tfrac{m}{2i\hbar}[[\hat{O},\hat{q}],\gamma\dot{\hat{q}}(t)]_{+}\end{aligned} \tag{4}$$

It is generally convenient to express the Markov Quantum langevin equation in terms of dimensionless position operator, $\hat{Q} = \frac{\hat{a}+\hat{a}^{\dagger}}{\sqrt{2}}$ and the dimensionless momentum fluctuations $\hat{P}_{in}(t)$.

$$\begin{aligned}\dot{\hat{O}} = \tfrac{1}{i\hbar}[\hat{O},\hat{H}_{sys}] + i\sqrt{2\gamma}[\hat{O},\hat{Q}]\hat{P}_{in}(t) \\ +\tfrac{1}{2iQ}[[\hat{O},\hat{Q}],\dot{\hat{Q}}(t)]_{+}\end{aligned} \tag{5}$$

where $Q = \Omega/\gamma$ is the oscillator's quality factor. Generally in quantum optics and quantum Optomechanics the bath coupling rate is much smaller than other relevant rate in the system ,in this case it is convenient to perform rotating wave approximation.Within the rotating wave approximation the Quantum langevin equation becomes,

$$\begin{aligned}\dot{\hat{O}} = \frac{1}{ih}[\hat{O},\hat{H}_{sys}] - [\hat{O},a^{\dagger}](\frac{\gamma}{2}a - \sqrt{\gamma}a_{in}(t)) \\ +(\frac{\gamma}{2}a^{\dagger} - \sqrt{\gamma}a_{in}^{\dagger}(t))[\hat{O},a]\end{aligned} \tag{6}$$

Here the Bath forcing term $F(t)$ is replaced by the input noise operator, $a_{in}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega e^{-i\omega(t-t')}a_{-}(\omega)$. Where $a_{-}(\omega)$ is the annihilation operator for the Bath oscillator with resonance frequency ω at some initial time $t_{-} < t$.

3 Quantum langevin equation for the quadrature operator

We have calculated the quantum langevin equation for the resonator quadrature operators using equation (6). The quantum Langevin equations for the Quadrature operators are,

$$\dot{X} = \Delta Y + \sqrt{2}g_0 QY + \epsilon\sqrt{2} + \frac{1}{\sqrt{2}}ig_e(\sigma_+ - \sigma_-) - \frac{k}{2}X + \sqrt{k}X_{in}$$
$$\dot{Y} = -\Delta X - \sqrt{2}g_0 QX + \epsilon\sqrt{2} - \frac{1}{\sqrt{2}}g_e(\sigma_+ + \sigma_-) - \frac{k}{2}Y + \sqrt{k}Y_{in}$$
$$\dot{Q} = \Omega_m P$$
$$\dot{P} = -\Omega_m Q - \frac{g_0}{\sqrt{2}}(X^2 + Y^2) - \sqrt{2}\lambda\sigma_z - \frac{\Gamma_m}{2}P + \sqrt{\Gamma_m}P_{in}$$
$$\dot{\sigma_z} = -2ig_c(\sigma_+ \frac{X + iY}{\sqrt{2}} - \sigma_- \frac{X - iY}{i\sqrt{2}}) - \frac{\Gamma_p}{2}(\sigma_z + \frac{1}{2})$$
$$\dot{\sigma_+} = i\Delta_q\sigma_+ i2\sqrt{2}\lambda Q\sigma_+ - ig_c\sigma_z \frac{X - iY}{\sqrt{2}} - \frac{\Gamma_p}{2}\sigma_+$$
$$\dot{\sigma_-} = -i\Delta_q\sigma_- - i2\sqrt{2}\lambda Q\sigma_- + ig_c\sigma_z \frac{X + iY}{\sqrt{2}} - \frac{\Gamma_p}{2}\sigma_-$$

Where k is the cavity field decay rate,Γ_m is the mechanical damping rate,Γ_r and Γ_p are the relaxation rate and dephasing rate of the quantum dot respectively.

4 Quantum Langevin Equation For The Fluctuating Operators

These equations are non linear coupled differential equations. To linearize them we can work in the steady state conditions (the steady state condition can be verified by having negative real roots of the drift matrix A) where the dynamics of the system can be described by the fluctuating operators which are fluctuating around their steady state value. We write all the operators used above as the sum of its steady state value and its fluctuations around its classical steady state value as follows.
$Q = Q_s + \delta Q, P = P_s + \delta P, X = X_s + \delta X, Y = Y_s + \delta Y, \sigma_+ = \sigma_s + \delta\sigma_+, \sigma_- = \sigma_s + \delta\sigma_-, \sigma_z = \sigma_z + \delta\sigma_z$
Where Q_s is the steady state value and δQ is the fluctuation. Similarly for

the other operators.The Langevin equations for the fluctuating operators are,

$$
\begin{aligned}
\delta\dot{X} &= \Delta_1\delta Y + \sqrt{2}g_0Y_s\delta Q + \frac{ig_c}{\sqrt{2}}(\delta\sigma_+ - \delta\sigma_-) - \frac{k}{2}\delta X + \sqrt{k}\delta X_{in} \\
\delta\dot{Y} &= -\Delta_1\delta X - \sqrt{2}g_0X_s\delta Q - \frac{g_c}{\sqrt{2}}(\delta\sigma_+ + \delta\sigma_-) - \frac{k}{2}\delta Y + \sqrt{k}\delta Y_{in} \\
\delta\dot{Q} &= \Omega_m\delta P \\
\delta\dot{P} &= -\Omega_m\delta Q - \sqrt{2}g_0(X_s\delta X + Y_s\delta Y) - \sqrt{2}\lambda\delta\sigma_z - \frac{\Gamma_m}{2}\delta P + \sqrt{\Gamma_m}\delta P_{in} \\
\delta\dot{\sigma_z} &= -2ig_c[(\sigma_s - \sigma_s^*)\frac{\delta X}{\sqrt{2}} - i(\sigma_s + \sigma_s^*)\frac{\delta Y}{\sqrt{2}} + \delta\sigma_+\frac{(X_s + iY_s)}{\sqrt{2}} - \delta\sigma_-\frac{(X_s - iY_s)}{\sqrt{2}}] - \frac{\Gamma_m}{2}\delta\sigma_z \\
\delta\dot{\sigma_+} &= i\Delta_2\delta\sigma_+ i2\sqrt{2}\lambda\sigma_s\delta Q - \frac{ig_c}{\sqrt{2}}(\sigma_z^s\delta X - i\sigma_z^s\delta Y + (X_s - iY_s)\delta\sigma_z) - \frac{\Gamma_p}{2}\delta\sigma_+ \\
\delta\dot{\sigma_-} &= -i\Delta_2\delta\sigma_- i2\sqrt{2}\lambda\sigma_s^*\delta Q + \frac{ig_c}{\sqrt{2}}(\sigma_z^s\delta X + i\sigma_z^s\delta Y + (X_s + iY_s)\delta\sigma_z) - \frac{\Gamma_p}{2}\delta\sigma_-
\end{aligned}
$$

Where $\Delta_1 = \Delta + \sqrt{2}g_0Q_s$, $\Delta_2 = \Delta_q + 2\sqrt{2}\lambda Q_s$ are the modified detuning of the cavity field and the quantum dot relative to the laser light. Here Q_s is the steady state displacement of the NMR.These equations can be written in compact form as,$\dot{u} = Au(t) + B(t)$.Where $u = [\delta X, \delta Y, \delta Q, \delta P, \sigma_z, \sigma_+, \sigma_-]^T$ and $B)(t)$ is the noise vector given by,
$B(t) = [\sqrt{k}\delta X_{in}, \sqrt{k}\delta Y_{in}, 0, \sqrt{\Gamma_m}\delta P_{in}, 0, 0, 0]^T$. And A is the drift matrix given by,

$$
A = \begin{vmatrix}
-\frac{k}{2} & \Delta_1 & \sqrt{2}g_0Y_s & 0 & 0 & \frac{1}{\sqrt{2}}ig_c & \frac{1}{\sqrt{2}}ig_c \\
\Delta_1 & -\frac{k}{2} & -\sqrt{2}g_0X_s & 0 & 0 & -\frac{1}{\sqrt{2}}g_c & -\frac{1}{\sqrt{2}}g_c \\
0 & 0 & \Omega_m & 0 & 0 & 0 & 0 \\
-\sqrt{2}g_0X_s & -\sqrt{2}g_0Y_s & -\Omega_m & -\frac{\Gamma_m}{2} & -\sqrt{2}\lambda & 0 & 0 \\
-\sqrt{2}ig_c(\sigma_s - \sigma_s^*) & -\sqrt{2}g_c(\sigma_s + \sigma_s^*) & 0 & 0 & -\frac{\Gamma_m}{2} & -ig_c\sqrt{2}(X_s + iY_s) & -ig_c\sqrt{2}(X_s - iY_s) \\
-\frac{1}{\sqrt{2}}ig_c\sigma_z^s & \frac{1}{\sqrt{2}}g_c\sigma_z^s & i2\sqrt{2}\lambda\sigma_s & 0 & \frac{1}{\sqrt{2}}ig_c(X_s - iY_s) & i\Delta_2 - \frac{\Gamma_p}{2} & 0 \\
\frac{1}{\sqrt{2}}ig_c\sigma_z^s & \frac{1}{\sqrt{2}}g_c\sigma_z^s & -i2\sqrt{2}\lambda\sigma_s^* & 0 & \frac{1}{\sqrt{2}}ig_c(X_s + iY_s) & 0 & -i\Delta_2 - \frac{\Gamma_p}{2}
\end{vmatrix} \tag{7}
$$

Where the input noise operators $\delta X_{in}, \delta Y_{in}$ are given by,

$$\delta X_{in} = \tfrac{1}{\sqrt{2}}(\delta a_{in} + \delta a_{in}^\dagger) \text{ and } \delta Y_{in} = \tfrac{1}{i\sqrt{2}}(\delta a_{in} - \delta a_{in}^\dagger)$$

with $\langle\delta a_{in}\rangle = \langle\delta a_{in}^\dagger\rangle = 0$ and $\langle\delta a_{in}(t)\delta a_{in}^\dagger(t')\rangle = \delta(t - t')$, $\delta(t)$ is the dirac delta function.

The above equations are linear but still coupled differential equations. To uncouple them we have to diagonalize the drift matrix A. From the diagonalised matrix we can easily get the value of $\delta Q(t)$.
Now the position spectrum of the NMR for its fluctuating operator is given by,

$$S_{\delta Q\delta Q}(\omega) = \int_{-\infty}^{\infty} \langle \delta Q(t)\delta Q(0)\rangle e^{i\omega t} \tag{4.22}$$

Where $\langle \delta Q(t)\delta Q(0)\rangle$ is the position auto correlation function for the fluctuation operator. Integrating $S_{\delta Q\delta Q}(\omega)$ with respect to ω will give the variance in position i.e.

$$\langle \delta Q^2\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_{\delta Q\delta Q}(\omega)d\omega \tag{4.23}$$

Now this variance can be used to study its non classical properties,

$\frac{\langle \delta Q^2\rangle}{\langle \delta Q_0^2\rangle} < 1$; corresponds to Squeezing
$\frac{\langle \delta Q^2\rangle}{\langle \delta Q_0^2\rangle} > 1$; will lead to Amplifications.

Where δQ_0^2 is the variance for the zero point oscillation of the resonator.

5 conclusion

We tried to study the dynamics of a mechanical resonator coupled to a quantum dot within an optical cavity. We have derived the quantum Langevin equation for the quadrature operator of the mechanical resonator and the optical field. To linearize the derived equation we divided the quadrature operators into the sum of fluctuation operator and its steady state value and then calculated the langevin equation in fluctuation operators. Drift matrix A is calculated which can be diagonalized to uncouple the equations. From that we can calculate the position fluctuation $\delta Q(t)$ of the resonator.

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